SOLUTION OF THE PROBLEM OF A FLOW OF RAREFIED GAS OF CONSTANT DENSITY AROUND A SEMIINFINITE PLATE BY THE INTEGRAL DIFFUSION METHOD

A. T. Onufriev

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 7, No. 5, pp. 90-93, 1966

Several papers [1-4] have proposed approximate diffusion models which can be used to examine the transport process in a rarefied gas where the mean free path is large and transport is not determined by the local gradient of the particular quantity.

In this paper the integral diffusion model [2] is used to solve the problem of determination of the friction stress and velocity of a flow of an incompressible gas around a plane semi-infinite plate in the whole range of Knudsen numbers. The obtained solution is compared with published solutions and experimental data [9].

§1. The flow of a rarefied gas at constant density, velocity of sound, and mean free molecular path in the boundary layer at a plane semi-infinite plate is described by the system of equations [2]

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \mu \frac{\partial^2 \varphi}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{4}{3}\Lambda \frac{\partial \tau}{\partial y} = \rho c (u - \varphi),$$

$$\tau = -\frac{\rho c \Lambda}{3}\frac{\partial \varphi}{\partial y} = -\mu \frac{\partial \varphi}{\partial y}$$

with boundary conditions

$$y = 0, \quad v = 0, \quad \frac{\varphi}{2} = \frac{2-\sigma}{\sigma}\frac{\Lambda}{3}\frac{\partial \varphi}{\partial y},$$

$$y - \infty \qquad u = u_0$$

and initial condition $x = 0$; $u = u_0$.

Here σ is the coefficient of diffuse reflection.

On introducing the dimensionless variables

$$u' = \frac{u}{u_0}, \quad v' = \frac{v}{u_0 \beta}, \quad x' = \beta \frac{3x}{2\Lambda}, \quad y' = \frac{3y}{2\Lambda},$$

$$\varphi' = \frac{\varphi}{u_0}, \quad \beta = \frac{1}{M}\left(\frac{2}{\pi\gamma}\right)^{1/2} = \frac{3\mu}{2\Lambda\rho u_0}$$

the system of equations takes the form

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = \frac{\partial^2 \varphi'}{\partial y'^2},$$

$$\frac{\partial^2 \varphi'}{\partial y'^2} = \varphi' - u', \qquad \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \qquad (1.1)$$

with boundary conditions

$$y' = 0, \quad v' = 0, \quad \varphi' = A\frac{\partial \varphi'}{\partial y'} \quad \left(A = \frac{2-\sigma}{\sigma}\right),$$

$$y' = \infty, \qquad u' = 1,$$

and initial condition

$$x' = 0, \qquad u' = 1.$$

The friction stress at the wall is given by the expression

$$\tau(0, x') = -\frac{\rho c u_0}{2}\frac{\partial \varphi'}{\partial y'} = \frac{\rho c}{2A}\varphi(0, x')$$

or

$$c_f M = \frac{2\sigma}{2-\sigma}\left(\frac{2}{\pi\gamma}\right)^{1/2}\varphi'(0, x'), \quad x' = \beta\frac{3x}{2\Lambda} = \frac{2}{\pi\gamma}\frac{R}{M^2} = z^2.$$

Here γ is the ratio of specific heats, R is the Reynolds number, and in this case

$$\frac{2}{\pi\gamma} = 0.383, \quad \frac{R^{1/2}}{Mz} = 1.61 \quad \text{for } \gamma = \frac{5}{3}$$

$$\frac{2}{\pi\gamma} = 0.456, \quad \frac{R^{1/2}}{Mz} = 1.48 \quad \text{for } \gamma = \frac{7}{5}.$$

Elimination of φ' from the system of equations leads to an equation for the velocity u' of the gas

$$u'\frac{\partial u'}{\partial x'} - \left(\int_0^{y'}\frac{\partial u'}{\partial x'}dy'\right)\frac{\partial u'}{\partial y'} = \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial u'}{\partial x'}\frac{\partial^2 u'}{\partial y'^2} +$$

$$+ u'\frac{\partial^3 u'}{\partial x' \partial y'^2} - \frac{\partial u'}{\partial y'}\frac{\partial^2 u'}{\partial x' \partial y'} - \left(\int_0^{y'}\frac{\partial u'}{\partial x'}dy'\right)\frac{\partial^3 u'}{\partial y'^3} \qquad (1.2)$$

with boundary conditions

$$y' = 0, \quad u' + u'\frac{\partial u'}{\partial x'} = A\left[\frac{\partial u'}{\partial y'} + u'\frac{\partial^2 u'}{\partial x' \partial y'}\right],$$

$$y' = \infty, \quad u' = 1$$

and initial condition

$$x' = 0, \qquad u' = 1.$$

§ 2. The equation for u' was solved by the method of finite differences in the variables z and $\zeta = \ln(y' + \Delta)$. When the variable ζ is introduced the equation takes the form

$$u'\left[\frac{\partial u'}{\partial x}\right] - \left(\int_{\zeta_0}^{\zeta}\left|\frac{\partial u'}{\partial x'}\right| e^{\zeta} d\zeta\right) e^{-\zeta}\frac{\partial u'}{\partial \zeta} =$$

$$= \left[\frac{\partial u'}{\partial x'}\right] e^{-2\zeta}\left(\frac{\partial^2 u'}{\partial \zeta^2} - \frac{\partial u'}{\partial \zeta}\right) + e^{-2\zeta}\left[\frac{\partial^2 u'}{\partial \zeta^2} - \frac{\partial u'}{\partial \zeta}\right] +$$

$$+ u' e^{-2\zeta}\left[\frac{\partial^3 u'}{\partial x' \partial \zeta^2} - \frac{\partial^2 u'}{\partial x' \partial \zeta}\right] - e^{-2\zeta}\frac{\partial u'}{\partial \zeta}\left[\frac{\partial^2 u'}{\partial \zeta \partial x'}\right] -$$

$$- e^{-3\zeta}\left(\frac{\partial^3 u'}{\partial \zeta^3} - 3\frac{\partial^2 u'}{\partial \zeta^2} + 2\frac{\partial u'}{\partial \zeta}\right)\int_{\zeta_0}^{\zeta}\left[\frac{\partial u'}{\partial x'}\right] e^{\zeta} d\zeta$$

with boundary conditions

$$\zeta_0 = \ln \Delta, \quad u' + u'\frac{\partial u'}{\partial x'} = Ae^{-\zeta}\left[\frac{\partial u'}{\partial \zeta} + u'\frac{\partial^2 u'}{\partial x' \partial \zeta}\right],$$

$$\zeta = \infty; \qquad x' = 0, \qquad u' = 1.$$

The right-hand side consists of terms, the differences for which were written with the (n + 1)-th layer included (n is the number of the point along x'). It must be taken into account that one of the characteristics of system (1.1) in the initial cross section is horizontal. The scheme of [10] is used to write the difference equation.

The friction stress is determined by the screw die method from the equation

Table

	$\sigma = 0.1$		$\sigma = 0.8$	
z	$u'(0, z)$	$\varphi'(0, z)$	$u'(0, z)$	$\varphi'(0, z)$
0	1.0	0.5	1.0	0.6
0.1	0.996	0.499		
0.2	0.983	0.496		
0.3	0.959	0.490		
0.4	0.928	0.481		
0.6	0.836	0.457		
0.8	0.719	0.424		
1.0	0.590	0.383	0.675	0.489
2.0	0.216	0.202	0.316	0.295
3	0.128	0.125	0.194	0.190
4	0.0904	0.0898	0.138	0.137
6	0.0576	0.0575	0.0872	0.0870
8	0.0424	0.0424	0.0640	0.0640
10	0.0337	0.0337	0.0507	0.0507
20	0.0166	0.0166	0.0249	0.0249
30	0.0111	0.0111	0.0166	0.0166

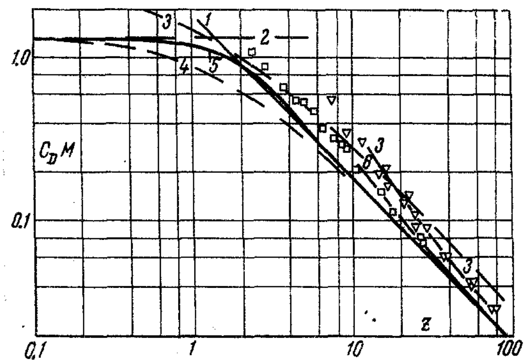

Fig. 1

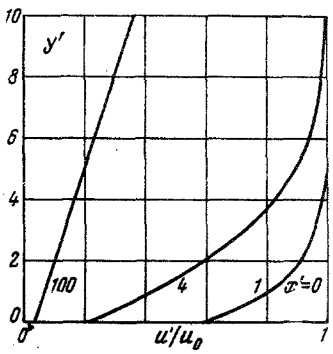

Fig. 2

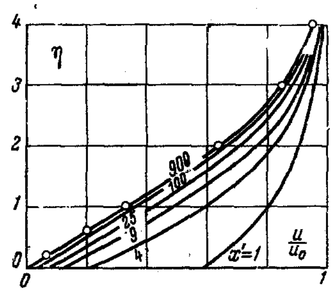

Fig. 3

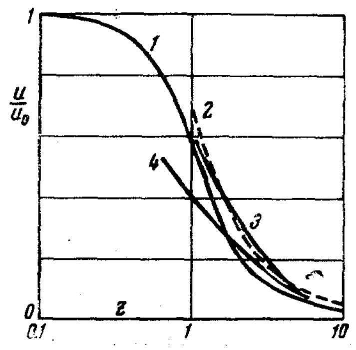

Fig. 4

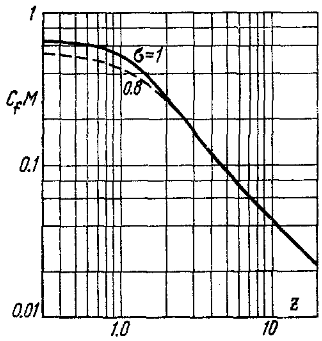

Fig. 5

$$\partial^2\varphi' / \partial y'^2 = \varphi' - u'$$

with boundary conditions

$$y' = 0, \quad \varphi' = A\frac{\partial\varphi'}{\partial y'}; \qquad y' = \infty, \quad \varphi' = 1.$$

§ 3. We consider the results of the calculations. The intervals were $\Delta\zeta = 0.05$ and 0.07; $\Delta z = 0.02$ and 0.01. A comparison of the errors with different pitches showed that the error at $x' = 1$ was ~0.3%.

The table gives the results for $u'(0, z)$ and $\varphi'(0, z)$ in relation to z for $\sigma = 1$ and $\sigma = 0.8$.

The results of different calculations were compared with the experimental data of [9] in [6]. This comparison is reproduced in Fig. 1, where the results of the present work are also given. The value of

$$C_D M = \frac{2}{x'}\int_0^{x'} C_f\, M\, dx' =$$

$$= 4\left(\frac{2}{\pi\gamma}\right)^{1/2}\frac{\sigma}{2-\sigma}\frac{1}{x}\int_0^{x'}\varphi'(0, x')\,dx'$$

for free molecular flow ($x' = 0$) is

$$C_{D_1}M = 2\sigma\sqrt{2/\pi\gamma},$$

$$\varphi(0, 0) = 0.5, \quad C_{D_1}M = 1.35 \quad \text{for} \quad \sigma = 1.0,$$

$$\varphi(0, 0) = 0.6, \quad C_{D_1}M = 1.08 \quad \text{for} \quad \sigma = 0.8.$$

In Fig. 1, curve 1 is the Blasius solution for an incompressible boundary layer; curve 2 is given by the theory of free-molecular flow; curve 3 is given by slip flow theory in Rayleigh's approximation [5]; curve 4 gives the results of calculations by the approximate method of [6]; curve 5 gives the results of calculations by the integral diffusion method; curve 6 is the relationship after introduction of a connection [7] for the finite length of the plate when $M = 0.60$ and; curve 7 is the same for $M = 0.18$. The experimental data of [9] are denoted by triangles for $0.16 < M < 0.21$, and by squares for $0.46 < M < 0.72$.

Figures 2 and 3 show the obtained velocity distribution in the cross section of the boundary layer at different distances from the front edge of the plate, while Fig. 4 shows the velocity distribution along the plate. In Fig. 4 curve 1 was obtained by calculation by the integral diffusion method for $\sigma = 1$; curve 2 is the same for $\sigma = 0.8$; curve 3 is given by the integral diffusion method in Rayleigh's approximation [2] and, curve 4 is given by slip flow theory in Rayleigh's approximation.

Figure 5 shows the distribution of $C_f M$ along the plate for $\sigma = 1.0$ and 0.8.

The obtained results agree with those of [8], where it was found that at $0.001 < M/R^{1/2} < 0.1$ the friction stress agreed exactly with the Blasius solution to terms of the third order of smallness.

REFERENCES

1. A. K. Timiryazev, "A simplified method of finding approximate solutions of the integral equation in the theory of internal friction of rarefied gases," Zh. eksperim. i teor. fiz., vol. 4, no. 5, 1934.

2. A. T. Onufriev, "A model of nonequilibrium processes in some problems of the mechanics of continuous media," PMTF, no. 1, 1963.

3. E. R. G. Eckert, Similarities between Energy Transport in Rarefied Gases and by Thermal Radiation; Modern Developments in Heat Transfer, ed. W. Ibele, Ac. Press, New York-London, 1963.

4. R. Goulard, A Milne-Eddington Model in Heat and Mass Transfer, International Symposium on Fundamental Phenomena in Hypersonic Flow, New York, 1964.

5. G. N. Patterson, Molecular Flow of Gases [Russian translation], Fizmatgiz, 1960.

6. V. A. Sukhnev, "Calculation of friction drag of a flat plate at zero angle of attack in the region between the continuum and free-molecular flow," Izv. AN SSSR, OTN, Mekhanika i mashinostroenie, no. 3, 1962.

7. Y. H. Kuo, "On the flow of an incompressible viscous fluid past a flat plate at moderate Reynolds numbers," J. Math. and Phys., vol. 32, no. 2-3, 1953.

8. V. P. Shidlovskii, "Consideration of slip in the flow of a viscous gas around a semiinfinite flat plate," Izv. AN SSSR, OTN, no. 9, 1958.

9. S. A. Shaaf and F. S. Sherman, "Skin friction in slip flow," J. Aeronaut. Sci., vol. 21, no. 2, 1954 (Russian translation in: Mechanics. A Collection of Translations and Reviews of Foreign Periodical Literature, no. 1, 1955.).

10. G. S. Glushko, "The turbulent boundary layer on a flat plate with an incompressible fluid," Izv. AN SSSR, Mekhanika, no. 4, 1965.

18 April 1966 Novosibirsk